Thermal Modeling of Microwave Percutaneous Hepatic Tumor Ablation

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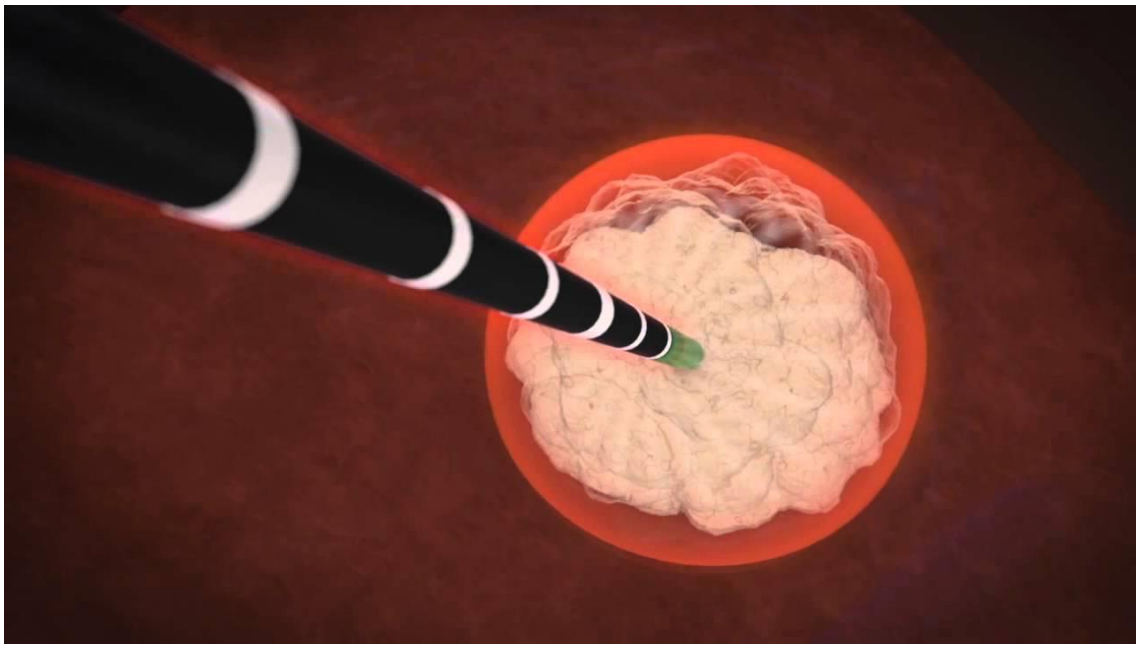

1. Background and Introduction

Thermal ablation is widely accepted as a method of treatment for certain benign or malignant tumors in the kidneys, liver, and bones [1, 2]. The most prevalent forms of thermal ablation include radiofrequency ablation (RFA) and microwave ablation (MWA), owing to their ability to generate heat in the tissue and raise the tissue temperature to the lethal point, 50-60 °C [2]. This temperature range is known to induce coagulation necrosis, i.e. cell death. The principle mechanisms behind RFA and MWA are the Joule effect and dielectric heating, respectively [2]. Briefly, according to the Joule effect, or Joule's first law, RF current causes resistive heat in the electric conductive tissue, whereas MWA introduces electromagnetic field to create molecular motions, generating heat due to the dielectric property of the tissue [2]. MWA is shown to be more promising in treating tumors, especially for those who have a tumor diameter greater than 3 cm and for tissues that have high impedance which prevents RF current flow, because of its capability to rapidly generate heat and to ignore tissue impedance [1, 2, 3].

A typical percutaneous surgical procedure for MWA involves insertion of an antenna, which acts as the applicator, into the tumor area with image guidance. A power generator supplies a power of around 0-300W depending on the number of antennae employed and the frequency used [1, 2, 3]. Specifically, there are three types of MWA: i) First generation without a coupled cooling system; ii) Second generation with a coupled cooling system but limited magnitude of power; iii) Third generation with both cooling and high power generation abilities [1]. The temperature profile, or the ablation zone, is primarily influenced by the tissue properties and the microwave interaction with the tissue [3].

2. Problem Statement

Hepatic tumors are a common target of MWA, and a theoretical model could enable us to gain an insight into the heat transfer behavior to further study the relationship between propagation of the electromagnetic wave and heat transfer in the liver tissue [3]. The aim of this project is to model a 1-Dimensional temperature profile within hepatic tumor tissue from the tip of the applicator to 1 cm beneath the tissue to justify the aforementioned heat conduction and ablation efficiency with respect to time and space with given tissue properties, ablation system properties, an initial condition, and boundary conditions.

3. Relevant Mathematical Background

Microwave ablation is an attractive technology for clinical application because of fast and high heat delivery to target tissue and, in some cases, no contact requirement. Microwave generators deliver electromagnetic energy to target tissues through an antenna probe at frequencies of 915 Mhz or 2.45 Ghz. These high electromagnetic frequencies induce rotation of molecules, such as water and proteins, rapidly generating heat in tissue. This process is called dielectric heating and is the fundamental transformation of energy in this system [3,4,6].

For the effective treatment of clinically observed liver tumors, the microwave ablation device must create a uniform heating area that can extend past the boundaries of the malignancy [3].

In this report, the goal is to model the temperature of tissue in a region-of-interest undergoing microwave ablation. A faithful representation of this medical treatment will describe the interaction of electromagnetic energy with liver tissue and the diffusion of delivered-heat through targeted tissue.

Microwave energy is characterized by Maxwell's equations [3].

$$\nabla \cdot \boldsymbol{D} = \rho_{free}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

D [C/m2] electric flux density
B [T] magnetic field
E [V/m] electric field strength
H [A/m] magnetic field intensity
ρ_{free} [C/m2] free charge density
J [A/m2] current density

The Maxwell equations can then be solved to determine the propagation of electromagnetic energy through a region of interest. The interactions of these electromagnetic fields with biological tissue can then be characterized using tissue properties such as density, specific heat, permittivity, and conductivity [6]. Materials that do not effectively absorb electromagnetic energy are referred to as *low-loss*, whereas other materials may exhibit high absorption of electromagnetic energy. This absorptivity can be defined as the material's conductivity, σ, divided angular frequency, ω, and dielectric permittivity, ε [3,4].

$$\varepsilon_r = \varepsilon'_r - \frac{j\sigma}{\omega\varepsilon_0}$$

Since most biological tissue is highly absorbing of propagating electromagnetic energy, its permittivity can be described with consideration of the electromagnetic frequency. The Cole-Cole model describes tissue permittivity as a function of frequency and tissue property constants [3].

$$\varepsilon(f) = \varepsilon_\infty - \frac{\Sigma(\varepsilon_s - \varepsilon_\infty)}{1 + (j2\pi f\tau_n).^{1-\alpha_n}} + \frac{\sigma_i}{2\pi\varepsilon_0}$$

ε_∞ permittivity at infinite frequency
ε_s permittivity at dc
f frequency
τ_n relaxation time constant
α attenuation constant
σ_i [S/m] dc conductivity

The conductivity and permittivity of tissue, like most biological systems, is dependent upon temperature itself. Temperature dependence of tissue dielectric properties arises primarily from the significant water concentration in organ tissue [3,6]. As microwave ablation heats an area of tissue, water molecules evaporate and rising temperatures irreversibly change protein structures, causing the conductivity and permittivity of the tissue to change [4].

To describe the heat diffusion in tissue, the Pennes bioheat equation is commonly employed.

$$\rho c \frac{dT}{dt} = \nabla \cdot k \nabla T + Q_{MW} - Q_p + Q_m$$

ρ [$\frac{kg}{m^3}$] mass density
c [$\frac{J}{kgK}$] specific heat
k [$\frac{W}{mK}$] thermal conductivity
T [K] temperature
Q [$\frac{W}{m^3}$] absorbed EM energy
Q_p[$\frac{W}{m^3}$] heat loss blood profusion
Q_m[$\frac{W}{m^3}$] metabolic heat generation

$$Q_p = \omega_{bl} c_{bl} (T - T_{bl})$$

ω_{bl} [$\frac{kg}{m^3 s}$] blood profusion rate
c_{bl} [$\frac{J}{KgK}$] specific heat capacity blood
T_{bl} [K] blood temperature

The heat source component Q_{MW} is generated in tissue by the electromagnetic energy absorption. This equation is called the specific absorption rate, SAR and describes the microwave heat source term in this system as a function of conductivity and electric field [3].

$$Q = \frac{1}{2}\sigma|E|^2$$

Now that the overall heat equation is defined and so is the thermal conductivity of tissue with respect to temperature, the focus can shift to the final piece of the SAR equation – the electric field component. This report has previously discussed how microwave dynamics can be described in terms of an electric field using the Maxwell equations. Therefore, to begin derivation of an electric field equation that is representative of microwave energy in this system, one can examine device constants.

A typical clinical microwave ablation device operates with 100 Watts of power at a frequency of 2.4 GHz.

$$P = I * V \quad [Watts = \frac{Joules}{sec}] \qquad f = 2.4\ GHz \qquad \omega = 2\pi f$$

From the power formula, current can be defined in terms of power and voltage. The current can then be substituted into the charge equation, q:

$$q = I * t = I_m \sin(\omega t) * t$$

$$q = \frac{P}{V_m \sin(\omega t)} t = \frac{\varepsilon_0 A}{2d} V_m \sin(\omega t)$$

Charge q is now defined in terms of voltage and frequency. It can now be input into the electric field equation for a conducting sphere:

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \quad [\frac{Volts}{m}]$$

Substitution of the q term into the spherical conductor equation produces an electric field equation that is dependent on source voltage, frequency, and distance from source [3,4,6].

$$E = \frac{V_m \sin(\omega t) d}{2r^2} \quad [\frac{Volts}{m}]$$

Considering that the final electric field equation for this microwave generator is dependent on voltage, one can work in reverse and solve for the electric field in terms of power.

$$E^2 = \frac{P}{A\varepsilon_0 d} t$$

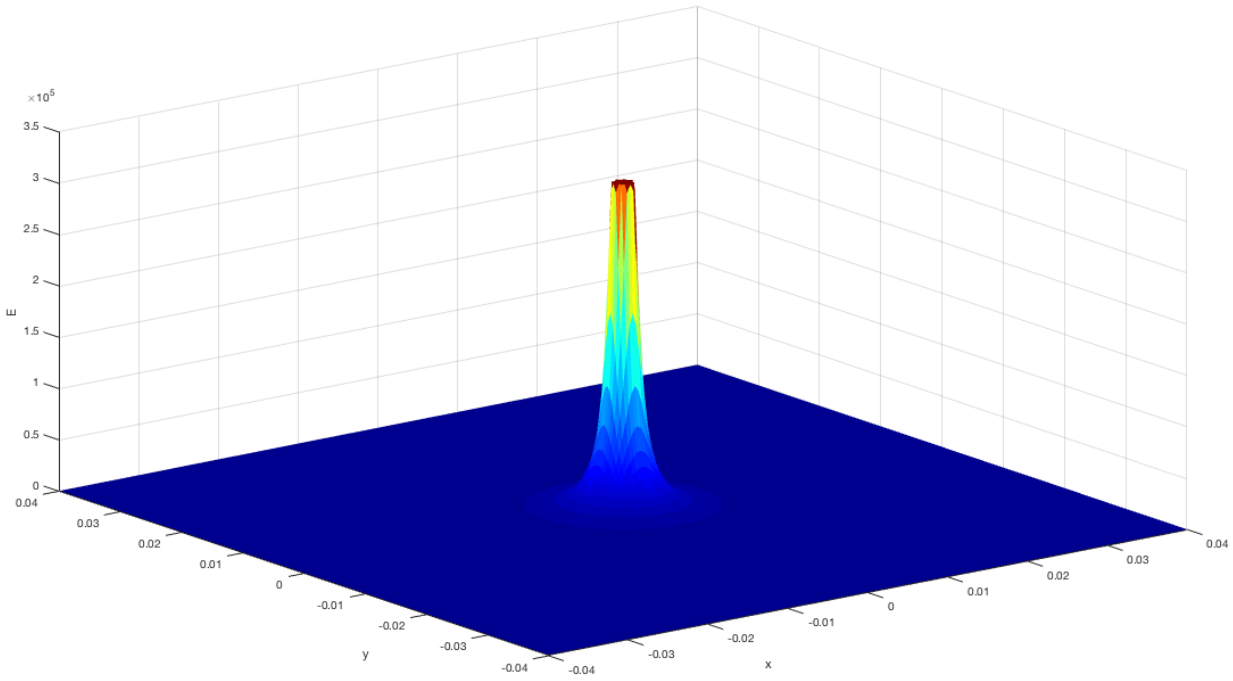

Figure 1. The surface plot above represents the intensity of the electric field generated by microwave system in this model. Notice that intensity is very high in a 2mm radius surrounding the ablation probe and falls off sharply thereafter. The spatial distribution visualized here is characteristic of any electric field governed by the Inverse-Square Law.

Considering the electric field dynamics described in this section, it is apparent that appropriate antenna designs must be incorporated into any effective clinical microwave ablation device to deliver adequate energy. The electric filed is crucial to the dielectric hysteretic heating observed in tissue ablation and enables microwave devices to have superior targeting and temperature control. Similar modeling approaches can be used for representing the electric and magnetic fields in RF-ablation and electrocautery.

Table 1. Constants relevant to heat diffusion for hepatic tissue and blood at 37°C. [4, 5]

Constants at 37°C	**Hepatic Tissue**	**Blood**
***P* (density) [kg/m^3]**	1060	1000
***c* (specific heat) [J/kg*K]**	3600	4180
***w* (perfusion) [1/s]**	*NA*	0.0064
***k* (thermal conductivity) [W/m*K]**	0.512	0.67
***σ* (electrical conductivity) [S/m]**	1.05 (but a function of T as temperature changes) [6]	0.667

4. Mathematical Solution

4.1 Simplified Equation Solution

Initially, an analytical solution to the full bio-heat equation was attempted. The solution that was arrived at and presented, however, was later realized to not be accurate. The nature of the full bio-heat equation makes an analytical solution prohibitively difficult to attain. With this in mind, a simplified version of the system was formulated.

In this simplified system, the heat provided by the microwave antenna is not represented by a source term, but by a constant boundary temperature, T_M. Many ablation systems regulate the temperature of the antenna to a constant level, so this is a reasonable assumption [1]. A further simplification of this model is that temperature change due to perfusion is also ignored. This should be kept in mind when viewing the results, as perfusion would normally be pushing the overall temperature of the system towards normal body temperature, T_o. As a result of these simplifications, the equation that we set out to solve analytically was the following:

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \quad \text{With Initial and Boundary conditions: } \begin{cases} T(x,0) = T_o \\ T(0,t) = T_M \\ T(L,t) = T_o \end{cases}, \text{ and } D = \frac{k}{\rho_L c_L}$$

where D is a constant determined by k, the heat diffusion coefficient; ρ_L, the density of liver tissue; and c_L, the specific heat of liver tissue. The initial condition sets the system at regular body temperature, T_o=37°C. The left boundary condition is T_M =100°C, as discussed above, and the right boundary condition is the end of the ablation zone, assumed to be a blood vessel, which provides a constant heat value of T_o. A length L of 4 cm is used, which corresponds to a 3cm tumor radius + 1cm buffer zone.

This is a standard heat equation with inhomogeneous boundary conditions. It was solved using the poison tooth extraction method. A homogeneous solution and a particular solution were found and added together to determine the full form of the solution to the equation. Once this was done, the constant A_n was found to complete the solution.

The first step was the homogeneous solution with boundary conditions equal to zero. This is solved for through the separation of variables method. This process is shown below.

Step 1, substitute the temperature function for two single-variable functions and rearrange:

$$T(x,t) = \phi(x)G(t)$$

$$\phi(x)\frac{\partial G(t)}{\partial t} = D\,\frac{\partial^2 \phi(x)}{\partial x^2}G(t)$$

$$\frac{\frac{\partial G(t)}{\partial t}}{G(t)} * \frac{1}{D} = \frac{\frac{\partial^2 \phi(x)}{\partial x^2}}{\phi(x)} = -\lambda$$

One half of this equation is dependent solely on *t*, while the other is dependent solely on *x*. Because the two halves are equal but dependent on different variables, they must both be equal to a constant, given by –λ.

Step 2, split the equation into two ODE halves and solve each independently.

Step 2.1, solving the time-dependent equation:

$$\frac{\frac{\partial G(t)}{\partial t}}{G(t)} * \frac{1}{D} = -\lambda$$

$$\frac{\partial G(t)}{\partial t} + \lambda D G(t) = 0$$

$$G(t) = e^{-\lambda D t}$$

Step 2.2, solving the space-dependent equation:

$$\frac{\frac{\partial^2 \phi(x)}{\partial x^2}}{\phi(x)} = -\lambda$$

$$\frac{\partial^2 \phi(x)}{\partial x^2} + \lambda \phi(x) = 0$$

$$\phi(x) = A cos\left(\sqrt{\lambda} x\right) + B sin(\sqrt{\lambda} x)$$

Step 3, substitute in the zero value boundary conditions to find λ:

$$\phi(0) = 0 = A sin(0) + B cos(0)$$

$$0 = B$$

$$\phi(L) = 0 = A sin\left(\sqrt{\lambda} L\right)$$

$$\sqrt{\lambda} L = n\pi \text{ Where } n = 1, 2, 3, \ldots$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\phi(x) = A_n \sin\left(\frac{n\pi}{L} x\right)$$

Step 4, recombine the two halves of equation and take their linear combination by the superposition principle for the homogeneous solution, $T_H(x,t)$:

$$T_H(x,t) = \phi(x)G(t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

With the homogeneous solution obtained, the next step is to solve for the particular solution. This is done by assuming steady state conditions and the original boundary conditions.

Step 5, determine the equation at steady state:

$$\frac{\partial T}{\partial t} = 0 \text{ (Steady state condition)}$$

$$0 = D\frac{\partial^2 T}{\partial x^2} \text{ With initial and boundary conditions: } \begin{cases} T(x,0) = T_o \\ T(0,t) = T_M \\ T(L,t) = T_o \end{cases}$$

Step 6, solve for $T_P(x)$:

$$T_P(x) = \int\int 0\, dx\, dx = c_1 x + c_2$$

$$T_P(0) = T_M = c_1(0) + c_2$$

$$c_2 = T_M$$

$$T_P(L) = T_o = c_1(L) + T_M$$

$$c_1 = \frac{T_o - T_M}{L}$$

$$T_P(x) = \frac{T_o - T_M}{L}x + T_M$$

Step 7, combine the particular and homogeneous solutions for the full solution:

$$T(x,t) = \frac{T_o - T_M}{L}x + T_M + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

Step 8, solve for the constant A_n using the equation for A_n

$$A_n = \frac{2}{L}\int_0^L \left(u(x,0) - T_P(x)\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$A_n = \frac{2}{L}\int_0^L \left(T_o - \frac{T_o - T_M}{L}x - T_M\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \int_0^L (T_o - T_M)\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_0^L \left(\frac{T_o - T_M}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

Here the integral is split into two parts. The first integral is simpler and will be solved first. The second integral requires integration by parts and will be solved afterwards.

Step 8.1, solve the first integral:

$$\frac{2}{L}\int_0^L (T_o - T_M)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2(T_o - T_M)}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2(T_o - T_M)}{L}\frac{-L}{n\pi}(\cos(n\pi) - \cos(0)) = \frac{-2(T_o - T_M)}{n\pi}((-1)^n - 1)$$

Step 8.2, solve the second integral using integration by parts:

$$\frac{2}{L}\int_0^L \left(\frac{T_o - T_M}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2(T_o - T_M)}{L^2}\int_0^L xsin\left(\frac{n\pi x}{L}\right)dx$$

$$u = x \quad dv = \sin\left(\frac{n\pi x}{L}\right)dx$$

$$du = dx \quad v = -\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}$$

$$\frac{2(T_o - T_m)}{L^2}\left[-xcos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi} - \int_0^L -\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}dx\right]$$

$$= \frac{2(T_o - T_m)}{L^2}\left[-xcos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi} + \sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right]_0^L$$

$$= \frac{2(T_o - T_m)}{L^2}\left[-\frac{L^2}{n\pi}cos(n\pi) + \sin(n\pi)\left(\frac{L}{n\pi}\right)^2 - 0 - \sin(0)\left(\frac{L}{n\pi}\right)^2\right]$$

$$= \frac{2(T_o - T_m)}{L^2}\left[-\frac{L^2}{n\pi}(-1)^n + 0 - 0 - 0\right]$$

$$= \frac{-2(T_o - T_m)}{n\pi}(-1)^n$$

Step 8.3, recombine the two integrals:

$$A_n = \frac{-2(T_o - T_M)}{n\pi}((-1)^n - 1) - \frac{2(T_o - T_m)}{n\pi}(-1)^n$$

$$A_n = \frac{-2(T_o - T_M)}{n\pi}((-1)^n - 1 - (-1)^n)$$

$$A_n = \frac{2(T_o - T_M)}{n\pi}$$

Step 9, assemble the full analytical solution:

$$T(x,t) = \frac{T_o - T_M}{L}x + T_M + \sum_{n=1}^{\infty} \frac{2(T_o - T_M)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

This solution was plotted in MATLAB. The resultant graph is shown in Figure 2 and Figure 3. The values used for the constants are given in Table 1. Note that temperatures near the center of the region, closest to the heat, reach lethal heat levels of 50+°C, but further from the center this drops off relatively quickly [2].

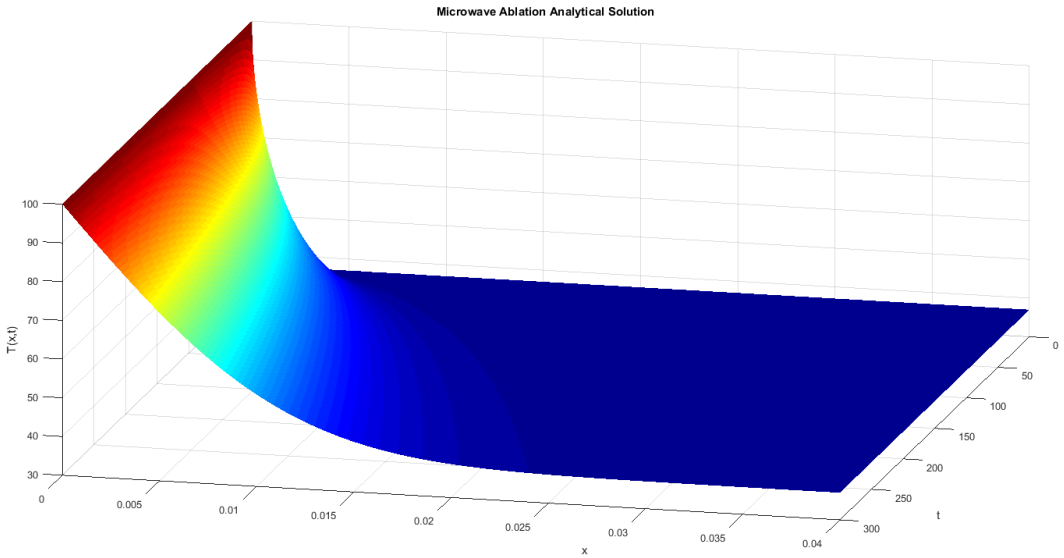

Figure 2. An isometric view of the analytical solution. The initial temperature at time zero is 37°C. The boundary temperature at x=0, where the heat is applied, is 100°C. Heat diffuses throughout the tissue as time increases.

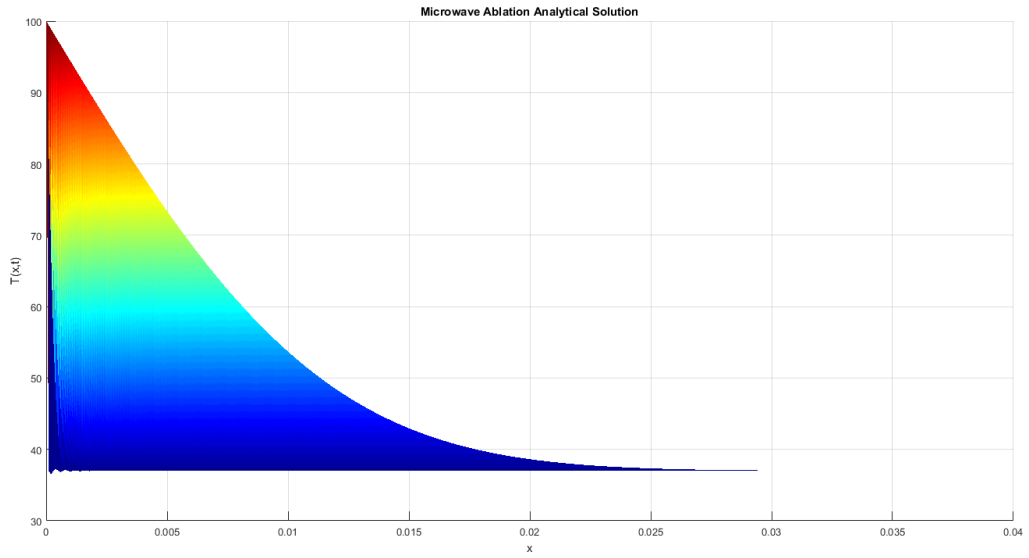

Figure 3. Another view of the analytical solution, showing the spatial temperature gradient.

To validate the analytical solution, a numerical solution to the equation was found using MATLAB's pdepe function. This result is shown below in Figure 4.

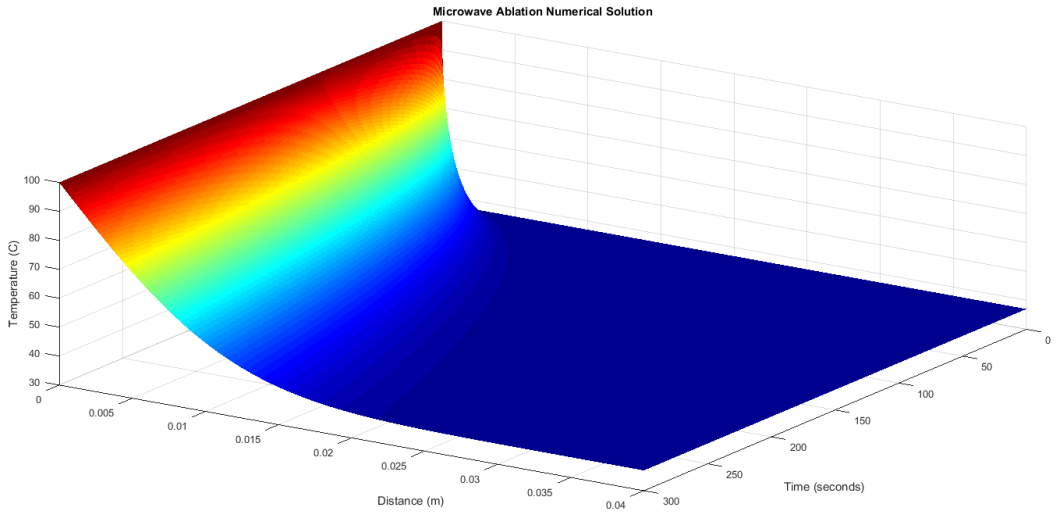

Figure 4. Numerical solution to the equation. This shows the same trend as the analytical solution, verifying that the solution is accurate.

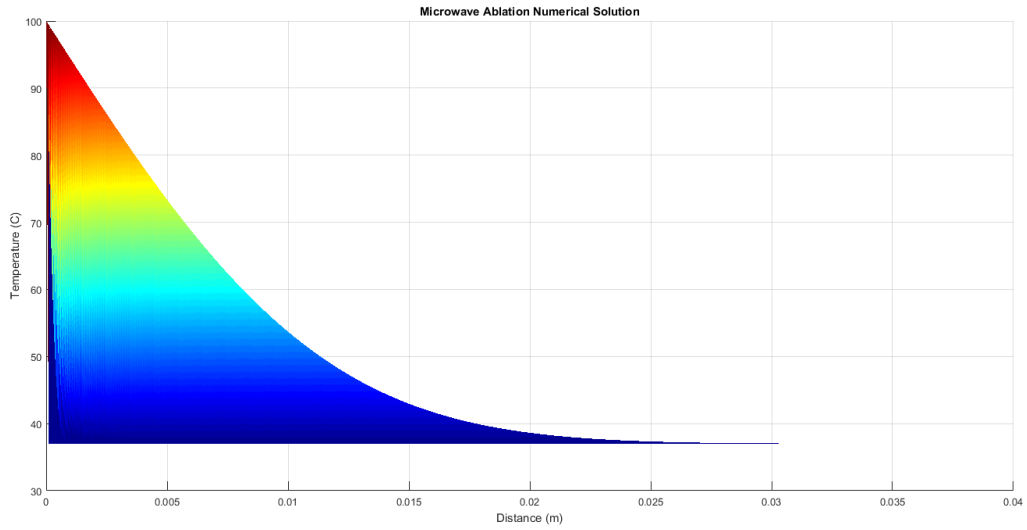

Figure 5. Numerical solution showing the spatial temperature gradient. Again, this is faithful to the analytical solution.

4.2 Full Solution

The simplified system provides an acceptable starting point to draw conclusions from, but it does not show the complete picture. In actuality, the electric field acts as a source term, providing energy to the whole system, not just the boundary. Additionally, the energy loss due to blood perfusion, which was ignored in the simplified equation, must be taken into account again. Furthermore, the tissue's electrical conductivity, σ, is not actually a constant, but a temperature-dependent variable [6]. With this information in mind we return to the original Pennes Bio-Heat Equation, given again below.

$$\rho_L c_L \frac{\partial T}{\partial t} = \nabla k \nabla T + Q_{MW} - Q_P + Q_m$$

Note that metabolic heat generation, Q_m is still negligible. The thermal conductivity k is constant with respect to x and the Q_P term represents the energy source of the microwave. Based on the literature research on electric field strength in MWA, the electric field can reasonably be modeled as a constant of 1000V/m [6]. With these specifics, the source terms can be defined as given below.

$$Q_{MW} = \frac{1}{2}|E|^2\sigma(T) = \frac{1}{2}|E|^2 * a_3\left[1 - \frac{1}{1+e^{a_1(a_2-T)}}\right] \text{, where } a_1 = 0.0697; a_2 = 85.375; a_3 = 2.173$$

$$Q_P = -\rho_b c_b w(T - T_o)$$

Note that the expression for *σ(T)* as well as the values for the constants a_1- a_3 are based on literature [6]. Substituting these into the Bio-Heat equation gives the full equation,

$$\rho_L c_L \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{1}{2}|E|^2 a_3 \left[1 - \frac{1}{1 + e^{a_1(a_2 - T)}}\right] - c_b w(T - T_o)$$

With initial and boundary conditions,

$$\begin{cases} T(x,0) = T_o \\ \dfrac{\partial T(0,t)}{\partial x} = 0 \\ T(L,t) = T_o \end{cases}$$

Note that the left boundary condition is different here from the simplified model. This zero flux condition indicates that the temperature is not changing at the location of the microwave antenna. This is valid because the high temperature and low surface area of microwave antennae used in ablation result in little heat flux into or out of the probe itself. The other two conditions are unchanged from the simplified equation.

Solving the full equation in MATLAB using pdepe gives the following results, shown in Figures 6, 7, and 8. The temperature reaches a maximum of 60°C, high enough to cause tissue death [2], and this level is sustained throughout the full region up until the border, which is cooled to the level of body temperature by a blood vessel (right boundary condition).

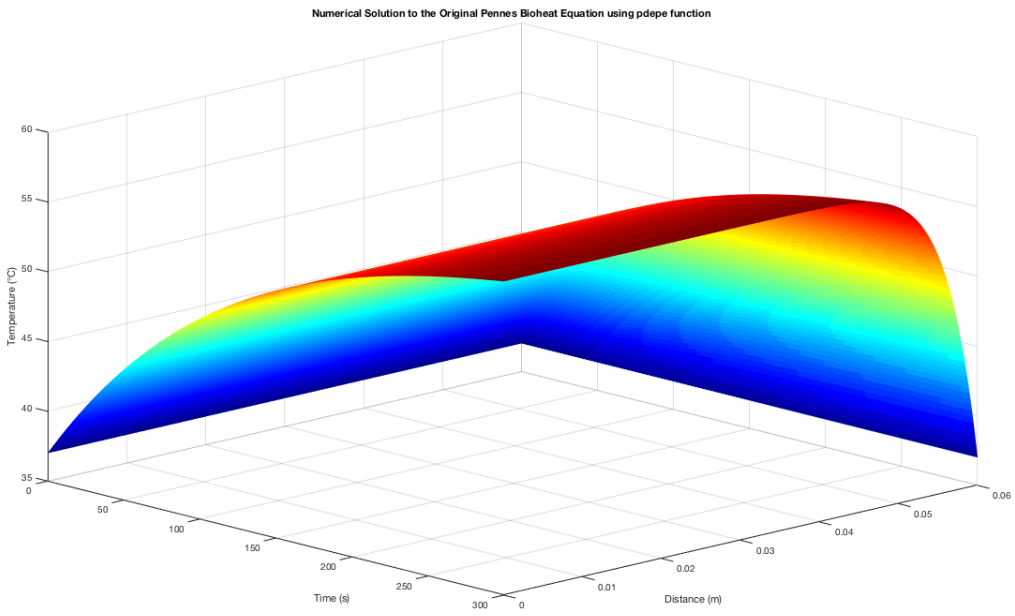

Figure 6. The diffusion of heat across time and space in the full model of hepatic tumor MWA.

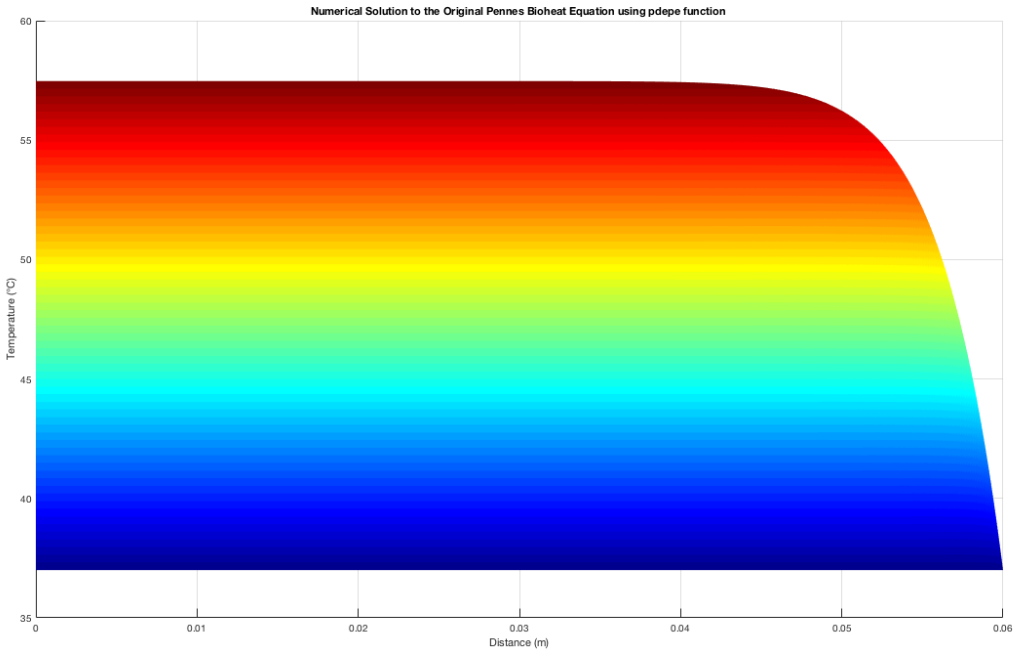

Figure 7. An alternate view of the full numerical solution, showing the temperature reached vs the distance from the microwave antenna.

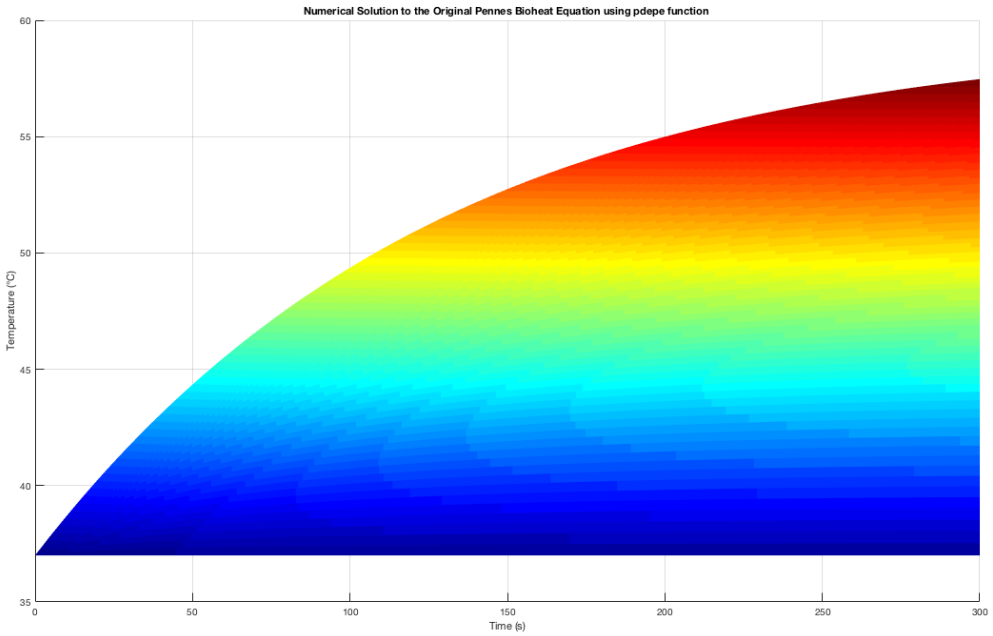

Figure 8. An alternate view of the full numerical solution, showing the temperature reached vs the duration of treatment.

5. Discussion

Due to the nonlinear nature of the tissue electric conductivity and the complexity of the blood perfusion equation in the governing partial differential equation, a simplified heat conduction model with modified boundary conditions was derived. As Figures 3 and 5 show, lacking of driving forces results in a decaying behavior of the temperature profile over the defined distance, and constant temperatures at the boundaries.

In contrast, the numerical solution to the full equation suggests that under the given initial and boundary conditions, a constant electric field and blood perfusion give rise to a saturating behavior as time proceeds as suggested in Figure 7. Moreover, Figure 8 shows that within the given time window, the temperature reaches the desired lethal point throughout the entire profile of the tumor tissue and drastically drops to body temperature near the far end boundary. These observations suggest that the model could serve as the theoretical heat diffusion model for the first generation MWA since there is no coupling cooling system to help prevent potential tissue damage due to heat near the far end of the boundary.

A suggested future model that matches the realistic situations should include the development of the system complexities, i.e. the interactions between microwave and the tissue, as well as the effect of a cooling agent.

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Appendix A: MATLAB Code for Simplified Analytical Solution

```
%% Microwave Thermal Ablation Analytical Solution
clear;
clc;

% Constants
L = 0.04; % meters
k = 0.512;%W/(m*C); thermal conductivity
rho_bl = 1000; %blood density; kg/m3
rho_liver = 1060;  %liver density; kg/m3
c_bl = 4100; %specific heat of blood J/(kg*C)
c_liver = 3600; %specific heat of liver J/(kg*C)
w = 0.0064; %Blood perfusion rate; 1/s
T_core = 37; %Core temperature; degree C
sig = 1.05; %electrical conductivity
T_m = 100; %boundary heat condition, provided by the microwave antenna

D = k/(rho_liver*c_liver);

xmesh = 0:.0001:L;
tmesh = 0:0.3:300;
[xx,tt]=meshgrid(xmesh,tmesh);


T_expansion = zeros(length(tmesh),length(xmesh)) + (T_core-T_m)/L.*xx+T_m; %
Poison Tooth Homogeneous + Particular

for n = 1:5000
    a_n = 2*(T_core-T_m)/(n*pi);
    T = a_n.*sin(n*pi.*xx/L).*exp(-D*(n*pi/L)^2.*tt);
    T_expansion = T_expansion + T;%T1+T2+T3;
end

figure(1)
surf(tt,xx,T_expansion,'EdgeColor','none')
%surf(tt,-xx,T_expansion,'EdgeColor','none')
xlabel('t')
ylabel('x')
zlabel('T(x,t)')
title('Microwave Ablation Analytical Solution')
colormap jet
```

Appendix B: MATLAB Code for Simplified Numerical Solution

```
%% Microwave Thermal Ablation Analytical Solution
clear;
clc;

global T_core T_m D
L = 0.04;
k = 0.512;%W/(m*C); thermal conductivity
rho_liver = 1060;  %liver density; kg/m3
c_liver = 3600; %specific heat of liver J/(kg*C)
T_core = 37; %Core temperature; degree C
T_m = 100;

D = k/(rho_liver*c_liver);

xmesh = 0:.0001:L;
tmesh = 0:0.3:300;
%[xx,tt]=meshgrid(xmesh,tmesh);
%%  Numerical Solution

sol_pdepe = pdepe(0,@pdefun,@ic,@bc,xmesh,tmesh);

figure(2)
hold on
surf(tmesh,xmesh,sol_pdepe','Edgecolor','none')
%surf(tmesh,-xmesh,sol_pdepe','Edgecolor','none')
xlabel('Time (seconds)')
ylabel('Distance (m)')
zlabel('Temperature (C)')
title('Microwave Ablation Numerical Solution')
grid on
hold off

function [c, f, s] = pdefun(x, t, u, DuDx)
    global D
    c = 1;
    f = D*DuDx;
    s = 0;
end

function u0 = ic(x)
    global T_core
    u0 = T_core;
end

function [pl,ql,pr,qr] = bc(xl,ul,xr,ur,t)
    global T_m T_core
    pl = ul-T_m;
    ql = 0;
    pr = ur-T_core;
    qr = 0;
end
```

Appendix C: MATLAB Code for Final Numerical Solution

```
%BENG 221 Project Microwave ablation of a hepatic tumor (numerical solution
to the original bioheat function)
close all; clc
%Constants
global D U  E a1 a2 a3 T_0 rho_liver c_liver
k = 0.512;%W/(m*K); thermal conductivity
rho_bl = 1000; %blood density; kg/m3
rho_liver = 1060;  %liver density; kg/m3
c_bl = 4180; %specific heat of blood J/(kg*K)
c_liver = 3600; %specific heat of liver J/(kg*K)
w = 0.0064; %Blood perfusion rate; 1/s
T_0 =37; %Initial temperature
a1 =0.0697; %a1-3 constants for electric conductivity constant
a2=85.375;
a3=2.173;
E = 800; % average electric field
xmesh =0:0.0001:0.06; %distance range in m
tmesh = 0:0.1:300; %time duration in s
D = k/(rho_liver*c_liver); % Rearranged diffusion constant
U = rho_bl*c_bl*w/(rho_liver*c_liver);
%% Plotting using pdepe
sol_pdepe = pdepe(0,@pdefun,@ic,@bc,xmesh,tmesh);
figure ()
surf(tmesh,xmesh,sol_pdepe', 'EdgeColor','none')
title('Numerical Solution to the Original Pennes Bioheat Equation using pdepe
function ')
xlabel('Time (s)')
ylabel('Distance (m)')
zlabel('Temperature (°C)')

%% Partial differential equation function
function [c, f, s] = pdefun(x, t, T, DTDx)
% PDE coefficients functions
global D U  a1 a2 a3 T_0 E rho_liver c_liver
c = 1;
f = D * DTDx;
%Driving forces, heat generation and blood perfusion
s = -U*(T-T_0)+ E^2/(2*rho_liver*c_liver)...
   *(a3*(1-1/(1+exp(a1*(a2-T)))));
end
% ---------------------------------------------------------------
%Initial condition function
function T0 = ic(x)

T0 = 37; %degree C
end
% ---------------------------------------------------------------
%Boundary conditions function
function [pl, ql, pr, qr] = bc(xl, Tl, xr, Tr, t)
pl = 0; % No flux left boundary condition
ql = 1;  % No flux left boundary condition
pr = Tr-37; % Constant rigt boundary condition
qr = 0;  % Constant rigt boundary condition
end
```